Übungsstunde 11:

Themen:

- Definitheit von Matrizen
- Singulärwertzerlegung (SVD) für die Ausgleichsrechnung

Definitheit: Hurwitz-Kriterium

A & A^T positiv definit genau dann, wenn

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1i} \\ \vdots & \ddots & \\ a_{i1} & & a_{ii} \end{bmatrix} > 0 \quad \forall i \qquad \text{in} \qquad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & & a_{nn} \end{bmatrix}$$

Bsp:

$$\underline{\underline{C}} = \begin{bmatrix} 2 & -3 & 7 \\ 4 & -5 & -4 \\ 3 & 6 & -1 \end{bmatrix}$$

$\Rightarrow \det(2) = 2 \boxed{>0}$

$$\det \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} = -10 + 12 = 2 \boxed{>0} \qquad \Rightarrow C \text{ positiv definit}$$

$$\det \begin{bmatrix} 2 & -3 & 7 \\ 4 & -5 & -4 \\ 3 & 6 & -1 \end{bmatrix} = 355 \boxed{>0}$$

Singulärwertzerlegung (SVD)

$$\underline{\underline{A}} = \underline{\underline{U}} \cdot \underline{\underline{S}} \cdot \underline{\underline{V}}^T \qquad A^{m\times n},\ U^{m\times m},\ S^{m\times n},\ V^{n\times n}$$

$$S = \begin{bmatrix} \hat{S} \\ 0 \end{bmatrix}$$

$$\|\underline{r}\|_2^2 = \|\underline{\underline{A}}\,\underline{x} - \underline{c}\|_2^2$$

$$= \|\underline{\underline{U}}\,\underline{\underline{S}}\,\underline{\underline{V}}^T\underline{x} - \underline{c}\|_2^2 \qquad U \,\&\, V \text{ sind orthogonal}$$

$$= \|\underline{\underline{U}}(\underline{\underline{S}}\cdot\underline{\underline{V}}^T\underline{x} - \underline{\underline{U}}^T\underline{c})\|_2^2$$

$$= \|\underbrace{\underline{\underline{S}}\,\underline{\underline{V}}^T\underline{x}}_{\begin{bmatrix} * \\ * \\ 0 \\ 0 \end{bmatrix}} - \underline{d}\|_2^2 \qquad \underline{\underline{U}}^T\cdot\underline{c} = \underline{d} = \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}$$

$$= \underbrace{\|\hat{\underline{\underline{S}}}\,\underline{\underline{V}}^T\underline{x} - d_0\|_2^2}_{=0} + \|d_1\|_2^2 = \|d_1\|_2^2 = \|r\|_2^2$$

$$\Rightarrow \hat{\underline{\underline{S}}}\,\underline{\underline{V}}^T\underline{x} = \underline{d_0}$$

$\hat{S}$ invertierbar: $\quad \underline{x} = \underline{\underline{V}}\,\hat{\underline{\underline{S}}}^{-1}\,\underline{d_0}$

$\hat{S}$ nicht " : $\quad \underline{x} = \underline{\underline{V}}\,\hat{\underline{\underline{S}}}^{+}\,\underline{d_0} \qquad \hat{S}^+$ Pseudoinverse

U: besteht aus den EV von AA^T

V: besteht aus den EV von A^TA

S: $\mathrm{diag}(\underbrace{\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_p}}_{\text{der Grösse nach geordnet}})$ EW von AA^T oder A^TA

$p = \min(m,n)$

$$\underline{v}^{(i)} = \frac{\underline{\underline{A}}^T\underline{u}^{(i)}}{s^{(i)}}, \qquad \underline{u}^{(i)} = \frac{\underline{\underline{A}}\,\underline{v}^{(i)}}{s^{(i)}}$$

Bsp: Prüfungsbeispiel HS 16

$$y = \beta_1 x + \beta_2$$

x_i	2	0	-2
y_i	$\sqrt{6}$	$-\sqrt{6}$	$2\sqrt{6}$

$$\underline{\underline{A}}\,\underline{\beta} = \underline{b} \qquad \underline{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -2 & 1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} \sqrt{6} \\ -\sqrt{6} \\ 2\sqrt{6} \end{bmatrix}$$

SVD:

1) U: AA^T: $\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 5 & 1 & -3 \\ 1 & 1 & 1 \\ -3 & 1 & 5 \end{bmatrix}$$

EW: $\det(A - \lambda I) \overset{!}{=} 0 = \det \begin{bmatrix} 5-\lambda^{+} & 1 & -3 \\ 1^{-} & 1-\lambda & 1 \\ -3^{+} & 1 & 5-\lambda \end{bmatrix}$

$$= (5-\lambda) \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 5-\lambda \end{bmatrix} - \det \begin{bmatrix} 1 & -3 \\ 1 & 5-\lambda \end{bmatrix}$$

$$-3 \det \begin{bmatrix} 1 & -3 \\ 1-\lambda & 1 \end{bmatrix}$$

$$= (5-\lambda)\left[(1-\lambda)(5-\lambda) - 1\right] - \left[5 - \lambda + 3\right] - 3\left[1 + 3 - 3\lambda\right]$$

$$= (1-\lambda)(5-\lambda)^2 - (5-\lambda) - (5-\lambda) - 3 - 12 + 9\lambda$$

$$(1-\lambda)(\lambda^2-10\lambda+25)$$

$$= -\lambda^3 + 10\lambda^2 - 25\lambda + \lambda^2 - 10\lambda + \cancel{25} - \cancel{25} + 2\lambda + 9\lambda$$

$$= \lambda\left[-\lambda^2 + 11\lambda - 24\right] = -\lambda\left[\lambda^2 - 11\lambda + 24\right]$$

$$= -\lambda(\lambda-8)(\lambda-3) \overset{!}{=} 0$$

$$\hat{S}$$

$$\lambda_1 = \underline{8}, \quad \lambda_2 = \underline{3}, \quad \lambda_3 = \underline{0}$$

$$\sigma_1 = \sqrt{8} = \underline{2\sqrt{2}}, \quad \sigma_2 = \sqrt{3}, \quad \sigma_3 = \sqrt{0} = \underline{0} \quad \Rightarrow S = \underline{\begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}}$$

EV: $(A-\lambda I)x = 0$

$\lambda_1 = 8$:

$$\left[\begin{array}{ccc|c} -3 & 1 & -3 & 0 \\ 1 & -7 & 1 & 0 \\ -3 & 1 & -3 & 0 \end{array}\right] \xrightarrow{G.} \left[\begin{array}{ccc|c} -3 & 1 & -3 & 0 \\ 0 & -20 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow x_3 = s, \quad x_2 = 0, \quad x_1 = -s$$

$$\Rightarrow E_8 = \operatorname{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\} = \underline{\operatorname{span}\left\{\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}}$$

$\lambda_2 = 3$:

$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & 0 \\ 1 & -2 & 1 & 0 \\ -3 & 1 & 2 & 0 \end{array}\right] \xrightarrow{G.} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 0 \\ -3 & 1 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{G.} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right] \xrightarrow{G.} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow x_3 = \underline{s}, \quad x_2 = \underline{s}, \quad x_3 = \underline{s}$$

$$\Rightarrow E_3 = \underline{\operatorname{span}\left\{\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}}$$

$\lambda_3 = 0$:
$$\left[\begin{array}{ccc|c} 5 & 1 & -3 & 0 \\ 1 & 1 & 1 & 0 \\ -3 & 1 & 5 & 0 \end{array}\right] \xrightarrow{G} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 4 & 8 & 0 \end{array}\right]$$
$$\xrightarrow{G} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \quad x_3 = \underline{s}$$
$$x_2 = \underline{-2s} \qquad \Rightarrow \quad E_0 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$
$$x_1 = \underline{s} \qquad\qquad = \underline{\operatorname{span}\left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}}$$

$$\Rightarrow \quad U = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$
$$= \frac{1}{\sqrt{6}} \begin{bmatrix} -\sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \\ \sqrt{3} & \sqrt{2} & 1 \end{bmatrix}$$

$$\underline{v}^{(i)} = \frac{\underline{\underline{A}}^T \underline{u}^{(i)}}{s^{(i)}}$$

$$\underline{v}^{(1)} = \frac{\underline{\underline{A}}^T \underline{u}^{(1)}}{\sigma^{(1)}}$$
$$= \frac{\begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}{2\sqrt{2}} = \frac{1}{4} \begin{bmatrix} -4 \\ 0 \end{bmatrix} = \underline{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}$$

$$\underline{v}^{(2)} = \frac{\begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{3}} = \frac{1}{3} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \underline{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$\Rightarrow \quad \underline{\underline{V}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{\underline{A}} = \underline{\underline{U}}\ \underline{\underline{S}}\ \underline{\underline{V}}^T$$

$$\underline{d} = \underline{\underline{U}}^T \underline{c} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{6} \\ -\sqrt{6} \\ 2\sqrt{6} \end{bmatrix} = \begin{bmatrix} \left.\begin{matrix} \sqrt{3} \\ 2\sqrt{2} \end{matrix}\right\} d_0 \\ * \end{bmatrix}$$

$$\hat{S} V^T x = d_0$$

$$\underline{x} = \underline{V}\ \hat{\underline{\underline{S}}}^{-1} \underline{d_0}$$

$$= \underline{V}\ \hat{\underline{\underline{S}}}^{+} \underline{d_0} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 2\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 2\sqrt{2} \end{bmatrix} = \underline{\underline{\begin{bmatrix} -\frac{\sqrt{3}}{2\sqrt{2}} \\ \frac{2\sqrt{2}}{\sqrt{3}} \end{bmatrix}}}$$

$$\hat{\underline{\underline{S}}} = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix}$$

$$\hat{S}^{-1} = \frac{1}{2\sqrt{6}} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix} = \underline{\begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix}}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$V: \quad A^T A = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\lambda_1 = 8 \;, \; \lambda_2 = 3 \qquad \Rightarrow \quad S = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$$

$$\sigma_1 = 2\sqrt{2} \;, \; \sigma_2 = \sqrt{3}$$

$$\lambda_1 = 8: \quad \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -5 & 0 \end{array}\right] \quad \Rightarrow \quad x_2 = 0 \;, \; x_1 = s$$

$$E_8 = \operatorname{span} \left\{ \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$$

$$\lambda_2 = 3: \quad \left[\begin{array}{cc|c} 5 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad E_3 = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\Rightarrow \quad V = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$u^{(i)} = \frac{A \, v^{(i)}}{\sigma^{(i)}} =$$

$$u^{(1)} = \ldots$$

$$u^{(2)} = \ldots$$

Zusammenhang u & v:

$$\underline{\underline{A}}^T \underline{\underline{A}}\, \underline{v} = \lambda \underline{v} \qquad \Big|\, \underline{\underline{A}}\cdot$$

$$(\underline{\underline{A}}\,\underline{\underline{A}}^T)\,\underline{\underline{A}}\,\underline{v} = \underline{\underline{A}}\lambda \underline{v} = \lambda \underline{\underline{A}}\,\underline{v} = \lambda \underline{u} \quad \Leftrightarrow \quad \underline{u} = \underline{\underline{A}}\,\underline{v}$$

$\underline{\underline{A}}\,\underline{v}$ ist ein EV von $\underline{\underline{A}}\,\underline{\underline{A}}^T$

$$\underline{\underline{A}}\,\underline{\underline{A}}^T \underline{u} = \lambda \underline{u} \qquad \Big|\, A^T\cdot$$

$$(\underline{\underline{A}}^T \underline{\underline{A}})\,\underline{\underline{A}}^T \underline{u} = \lambda\, \underline{\underline{A}}^T \underline{u} = \lambda \underline{v} \quad \Leftrightarrow \quad \underline{v} = \underline{\underline{A}}^T \underline{u}$$

$$\leadsto \quad \|\underline{v}\|_2 = \|\underline{\underline{A}}^T \underline{u}\|_2 = \|\underbrace{V}_{\text{orth.}} S^T \underbrace{U^T}_{\text{orth.}} \underline{u}\|_2 = \|\underline{\underline{S}}^T \underline{u}\|_2$$

$$= \|\underline{\underline{S}}\,\underline{u}\|_2 = \|s^{(1)} \underbrace{u^{(1)}}_{1}\|_2 = \|s^{(1)}\|_2$$

$$\Rightarrow \quad \underline{v}^{(i)} = \frac{\underline{\underline{A}}^T \underline{u}^{(i)}}{s^{(i)}}$$

dasselbe für $\quad \underline{u}^{(i)} = \dfrac{\underline{\underline{A}}\,\underline{v}^{(i)}}{s^{(i)}}$?

Übungsstunde 11:

Themen:

- Definitheit von Matrizen
- Singulärwertzerlegung (SVD) für die Ausgleichsrechnung

Definitheit: Hurwitz-Kriterium

$\underline{\underline{A}}$ ist positiv definit, falls:

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1i} \\ \vdots & \ddots & \\ a_{i1} & & a_{ii} \end{bmatrix} > 0 \quad \forall i \qquad \underline{\underline{A}} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & & a_{nn} \end{bmatrix}$$

Bsp:

$$\underline{\underline{C}} = \begin{bmatrix} 2 & -3 & 7 \\ 4 & -5 & -4 \\ 3 & 6 & -1 \end{bmatrix}$$

AA^T

A^TA

$$\Rightarrow \det[2] = 2 > 0 \checkmark$$

$$\det \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} = -10 - (-12) = 2 > 0 \checkmark$$

$$\det \begin{bmatrix} \ddots \end{bmatrix} = 355 > 0 \checkmark$$

? semi pos. def.

$<, >, <, >, \ldots$

$\leq, \geq, \leq, \geq, \ldots$

Singulärwertzerlegung (SVD)

$U S V^T = A$

$$\underline{\underline{A}} = \underline{\underline{U}}\,\underline{\underline{S}}\,\underline{\underline{V}}^T \qquad \underline{\underline{S}} = \begin{bmatrix} \hat{S} \\ 0 \end{bmatrix}, \; U \,\&\, V \text{ orthogonal}$$

$$\|\underline{r}\|_2^2 = \|\underline{\underline{A}}\,\underline{x} - \underline{c}\|_2^2 = \|\underline{\underline{U}}\,\underline{\underline{S}}\,\underline{\underline{V}}^T\underline{x} - \underline{c}\|_2^2$$

$$= \|\underline{\underline{U}}\,(\underline{\underline{S}}\,\underline{\underline{V}}^T\underline{x} - \boxed{\underline{\underline{U}}^T\underline{c}})\|_2^2$$

$$= \|\underline{\underline{S}}\,\underline{\underline{V}}^T\underline{x} - \underline{d}\|_2^2 = \underbrace{\|\underline{\underline{\hat{S}}}\,\underline{\underline{V}}^T\underline{x} - \underline{d_0}\|_2^2}_{=0} + \underbrace{\|\underline{d_1}\|_2^2}_{\text{Fehler}}$$

$$\Rightarrow \quad \underline{\underline{S}}\,\underline{\underline{V}}^T\underline{x} = \underline{d} \qquad \underbrace{\begin{bmatrix} \hat{S} \\ 0 \end{bmatrix} \underline{\underline{V}}^T\underline{x}}_{\begin{bmatrix} * \\ * \\ 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} d_0 \\ d_1 \end{bmatrix} \qquad \underline{d} = \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}$$

$$\begin{bmatrix} * \\ * \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}$$

$$\Downarrow$$

$$\underline{\underline{\hat{S}}}\,\underline{\underline{V}}^T\underline{x} = \underline{d_0}$$

$\hat{S}$ inv. $\Rightarrow$ $\underline{x} = \underline{\underline{V}}\,\underline{\underline{\hat{S}}}^{-1}\,\underline{d_0}$

$\hat{S}$ nicht " $\Rightarrow$ $\underline{x} = \underline{\underline{V}}\,\underline{\underline{\hat{S}}}^{+}\,\underline{d_0}$

Bsp: Prüfung HS 16

$$\underline{y = \beta_1 x + \beta_2}$$

x_i	2	0	-2
y_i	$\sqrt{6}$	$-\sqrt{6}$	$2\sqrt{6}$

$$\sum_{i=1}^{3} |y_i - (\beta_1 x + \beta_2)|^2$$

$$\underline{\underline{A}}\,\underline{\beta} = \underline{b} \qquad \underline{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -2 & 1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} \sqrt{6} \\ -\sqrt{6} \\ 2\sqrt{6} \end{bmatrix} = \sqrt{6} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\underline{\underline{A}} = \underline{\underline{U}}\,\underline{\underline{S}}\,\underline{\underline{V}}^T$$

$$A = 3 \begin{bmatrix} \cdot & \\ & \cdot \end{bmatrix}^{2} = S$$

U: EV von AA^T

V: EV von A^TA

S: $\mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_p})$ EW von AA^T o. A^TA, $p = \min\{m,n\}$

↑ Der Grösse nach sortiert

$$\underline{v}^{(i)} = \frac{\underline{\underline{A}}^T \underline{u}^{(i)}}{\sigma^{(i)}}, \quad \underline{u}^{(i)} = \frac{\underline{\underline{A}}\,\underline{v}^{(i)}}{\sigma^{(i)}}$$

$\underline{U}$:

$$\underline{\underline{A}}\,\underline{\underline{A}}^T = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -3 \\ 1 & 1 & 1 \\ -3 & 1 & 5 \end{bmatrix}$$

EW: $\det(\underline{\underline{A}}\,\underline{\underline{A}}^T - \lambda \underline{\underline{I}}) = \det \begin{bmatrix} 5-\lambda & 1 & -3 \\ 1 & 1-\lambda & 1 \\ -3 & 1 & 5-\lambda \end{bmatrix} \stackrel{!}{=} 0$

$$= (5-\lambda)\det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 5-\lambda \end{bmatrix} - \det\begin{bmatrix} 1 & -3 \\ 1 & 5-\lambda \end{bmatrix} - 3\det\begin{bmatrix} 1 & -3 \\ 1-\lambda & 1 \end{bmatrix}$$

$$= (5-\lambda)\left[(1-\lambda)(5-\lambda)-1\right] - \left[5-\lambda+3\right] - 3\left[1+3-3\lambda\right]$$

$$= (1-\lambda)\underbrace{(5-\lambda)^2}_{\lambda^2-10\lambda+25} - (5-\lambda) - (5-\lambda) - 3 - 3 - 9 + 9\lambda$$

$$= -\lambda^3 + 10\lambda^2 - 25\lambda + \lambda^2 - 10\lambda + \underbrace{25-25}_{=0} + 11\lambda$$

$$= -\lambda^3 + 11\lambda^2 - 24\lambda \stackrel{!}{=} 0$$

$$= -\lambda(\lambda^2 - 11\lambda + 24) = -\lambda(\lambda-8)(\lambda-3)$$

$$\Rightarrow \lambda_1 = \underline{8}, \quad \lambda_2 = \underline{3}, \quad \lambda_3 = \underline{0} \qquad \Rightarrow \underline{\underline{S}} = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \hat{S} \\ \}\, 0 \end{matrix}$$

$$\sigma_1 = \underline{2\sqrt{2}}, \quad \sigma_2 = \underline{\sqrt{3}}, \quad \sigma_3 = \underline{0}$$

EV:

$\lambda_1 = 8$:

$$\left[\begin{array}{ccc|c} -3 & 1 & -3 & 0 \\ 1 & -7 & 1 & 0 \\ -3 & 1 & -3 & 0 \end{array}\right] \xrightarrow{G} \left[\begin{array}{ccc|c} 1 & -7 & 1 & 0 \\ 0 & -23 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{matrix} x_3 = s \\ x_2 = 0 \\ x_1 = -s \end{matrix}$$

$$\Rightarrow E_8 = \operatorname{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\} = \operatorname{span}\left\{\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$\lambda_2 = 3:$
$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & 0 \\ 1 & -2 & 1 & 0 \\ -3 & 1 & 2 & 0 \end{array}\right] \xrightarrow{G.} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right]$$

$$\xrightarrow{G.} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_3 = s \\ x_2 = s \\ x_1 = s \end{array}$$

$$\Rightarrow E_3 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\} = \underline{\operatorname{span}\left\{\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}}$$

$\lambda_3 = 0:$
$$\left[\begin{array}{ccc|c} 5 & 1 & -3 & 0 \\ 1 & 1 & 1 & 0 \\ -3 & 1 & 5 & 0 \end{array}\right] \xrightarrow{G.} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 4 & 8 & 0 \end{array}\right]$$

$$\xrightarrow{G.} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_3 = s \\ x_2 = -2s \\ x_1 = s \end{array}$$

$$\Rightarrow E_0 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\} = \underline{\operatorname{span}\left\{\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}}$$

$$\Rightarrow \underline{\underline{U}} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \underline{\frac{1}{\sqrt{6}}\begin{bmatrix} -\sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \\ \sqrt{3} & \sqrt{2} & 1 \end{bmatrix}}$$

$$\underline{v}^{(1)} = \frac{\underline{\underline{A}}^T \underline{u}^{(1)}}{\sigma^{(1)}} = \frac{\begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}}{2\sqrt{2}} = \frac{1}{4} \begin{bmatrix} -4 \\ 0 \end{bmatrix} = \underline{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}$$

$$\underline{v}^{(2)} = \frac{\underline{\underline{A}}^T \underline{u}^{(2)}}{\sigma^{(2)}} = \frac{\begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{3}} = \frac{1}{3} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \underline{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$\Rightarrow \quad \underline{\underline{V}} = \underline{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\underline{\underline{A}}\,\underline{x} = \underline{\underline{U}}\,\underline{\underline{S}}\,\underline{\underline{V}}^T \underline{x} = \underline{b}$$

$$\Leftrightarrow \quad \underline{\underline{S}}\,\underline{\underline{V}}^T \underline{x} = \underline{\underline{U}}^T \underline{b} = \underline{d}$$

$$\Rightarrow \quad \hat{\underline{\underline{S}}}\,\underline{\underline{V}}^T \underline{x} = \underline{d}_0$$

$$\underline{d} = \underline{\underline{U}}^T \underline{b} = \begin{bmatrix} -\sqrt{3} & 0 & \sqrt{3} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1^{*} & -2^{*} & 1^{*} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ 2\sqrt{2} \\ 5 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \end{matrix}}\right\} d_0 \\ \left.\vphantom{a}\right\} d_1 = 5 \end{matrix}$$

$$\underline{x} = \underline{\underline{V}}\,\hat{\underline{\underline{S}}}^{-1} \underline{d}_0 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 2\sqrt{2} \end{bmatrix}$$

$$\hat{\underline{\underline{S}}}^{-1} = \begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \qquad = \begin{bmatrix} -\frac{1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 2\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\sqrt{3}}{2\sqrt{2}} \\ \frac{2\sqrt{2}}{\sqrt{3}} \end{bmatrix} \begin{matrix} = \mu_1 \\ \\ = \mu_2 \end{matrix}$$

$$\hat{S} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \qquad \hat{S}^+ = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 0 \end{bmatrix}$$

$\hat{S}^{-1}$ existiert nicht!

$$V: \quad A^T A = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -2 & 1 \end{bmatrix} = \underline{\begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix}}$$

$$\lambda_1 = \underline{8}, \quad \lambda_2 = \underline{3} \qquad \Rightarrow \underline{\underline{S}} = \underline{\begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}}$$

EV: $\lambda_1 = 8$: $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -5 & 0 \end{array}\right]$ $\quad x_2 = 0, \; x_1 = s \qquad E_8 = \underline{\text{Span}\left\{ \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}}$

$\lambda_2 = 3$: $\left[\begin{array}{cc|c} 5 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad x_2 = s, \; x_1 = 0 \qquad E_3 = \text{span}\left\{ \underline{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right\}$

$$\Rightarrow \underline{\underline{V}} = \underline{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\underline{u}^{(1)} = \frac{\underline{\underline{A}}\, \underline{v}^{(1)}}{\sigma^{(1)}} = \frac{\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix}}{2\sqrt{2}} = \frac{1}{2\sqrt{2}} \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{2}} \underline{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

$$\underline{u}^{(2)} = \frac{\underline{\underline{A}}\, \underline{v}^{(2)}}{\sigma^{(2)}} = \frac{\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\sqrt{3}} = \frac{1}{\sqrt{3}} \underline{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

$$\underline{\underline{U}} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & * \\ 0 & \frac{1}{\sqrt{3}} & * \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & * \end{bmatrix}$$

Zusammenhang U & V:

$$\underline{\underline{A}}^T \underline{\underline{A}}\, \underline{v}^{(i)} = \lambda\, \underline{v}^{(i)}$$

$$(\underline{\underline{A}}\,\underline{\underline{A}}^T)\, \underline{\underline{A}}\, \underline{v}^{(i)} = \lambda\, \underline{\underline{A}}\, \underline{v}^{(i)} = \lambda\, \underline{u}^{(i)}$$

$$\Leftrightarrow \underline{u}^{(i)} = \frac{\underline{\underline{A}}\, \underline{v}^{(i)}}{\sigma^{(i)}}$$

$$\underline{\underline{A}}\, \underline{\underline{A}}^T \underline{u}^{(i)} = \lambda\, \underline{u}^{(i)}$$

$$(\underline{\underline{A}}^T \underline{\underline{A}})\, \underline{\underline{A}}^T \underline{u}^{(i)} = \lambda\, \underline{\underline{A}}^T \underline{u}^{(i)} = \lambda\, \underline{v}^{(i)}$$

$$\Leftrightarrow \underline{v}^{(i)} = \frac{\underline{\underline{A}}^T \underline{u}^{(i)}}{\sigma^{(i)}}$$

$$\|\underline{\underline{A}}^T \underline{u}\|_2 = \|(\underline{\underline{U}}\,\underline{\underline{S}}\,\underline{\underline{V}}^T)^T \underline{u}\|_2 = \|\underline{\underline{V}}\,\underline{\underline{S}}^T \underline{\underline{U}}^T \underline{u}\|_2$$

$$= \|\underline{\underline{S}}^T \underline{u}\|_2$$

$$\|\underline{\underline{A}}\, \underline{v}\|_2$$

$$= \|\sigma^{(i)} \underline{u}^{(i)}\|_2 = \|\sigma^{(i)}\|_2$$